

# Representations of group rings and groups

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## Abstract

An isomorphism between the group ring of a finite group and a ring of certain block diagonal matrices is established. The group ring  $RG$  of a finite group  $G$  is isomorphic to the set of *group ring matrices* over  $R$ . It is shown that for any group ring matrix  $A$  of  $\mathbb{C}G$  there exists a matrix  $P$  (independent of the entries of  $A$ ) such that  $P^{-1}AP = \text{diag}(T_1, T_2, \dots, T_r)$  for block matrices  $T_i$  of fixed size  $s_i \times s_i$  where  $r$  is the number of conjugacy classes of  $G$  and  $s_i$  are the ranks of the group ring matrices of the primitive idempotents. Using the isomorphism of the group ring to the ring of group ring matrices followed by the mapping  $A \mapsto P^{-1}AP$  (where  $P$  is of course fixed) gives an isomorphism from the group ring to the ring of such block matrices. Specialising to the group elements gives a faithful representation of the group. Other representations of  $G$  may be derived using the blocks in the images of the group elements.

Examples are given demonstrating how interesting and useful representations of groups can be derived using the method.

For a finite abelian group  $Q$  an explicit matrix  $P$  is given which diagonalises any group ring matrix of  $\mathbb{C}Q$ . The matrix  $P$  is defined directly in terms of roots of unity depending only on an expression for  $Q$  as a product of cyclic groups. The characters and character table of  $Q$  may be read off directly from the rows of the diagonalising matrix  $P$ . This has applications to signal processing and generalises the cyclic group case.

## 1 Introduction

For background on groups and group rings, including information on conjugacy classes and representation theory, see [7], and for group ring matrices see [4]. Further information on representation theory and character theory may be found in [3] and/or [6]. Results are given over the complex numbers  $\mathbb{C}$  but many of the results hold over other suitably chosen fields.

A matrix  $A$  is said to be *diagonalised by*  $P$  if  $P^{-1}AP = D$  where  $D$  is a diagonal matrix. A circulant matrix can be diagonalised by the Fourier matrix of the same size. The diagonalising Fourier matrix is independent of the particular circulant matrix; this is the basis for the finite Fourier transform and the convolution theorem, see for example [2]. The Fourier  $n \times n$  matrix satisfies  $FF^* = nI_n$ , (and is thus a complex Hadamard matrix) and when the rows are labelled by  $\{1, g, g^2, \dots, g^{n-1}\}$ , it gives the characters and character table of the cyclic group  $C_n$  generated by  $g$ . The ring of circulant matrices over  $R$  is isomorphic to the ring of *group ring matrices* over  $R$  of the cyclic group, see for example [4].

The group ring of a finite group is isomorphic to the ring of group ring matrices as determined in [4]. The group ring matrices are types of matrices determined by their first rows; see section 2 below for precise formulation. For example circulant matrices are the group ring matrices of the cyclic group and matrices of the form  $\begin{pmatrix} A & B \\ B^T & A^T \end{pmatrix}$ , where  $A, B$  are circulant matrices, are determined by their first rows and correspond to the group ring matrices of the dihedral group. See Sections 2,4 and 5 below for further examples.

Group rings and group ring matrices will be over  $\mathbb{C}$  unless otherwise stated. Results may hold over other fields but these are not dealt with here.

An isomorphism from the ring of group ring matrices of a finite group  $G$  into certain block diagonal matrices is established. More precisely it is shown that for a group ring matrix  $A$  of a finite group  $G$  there exists a matrix  $P$  (independent of the particular  $A$ ) such that  $P^{-1}AP = \text{diag}(T_1, T_2, \dots, T_r)$  for

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block matrices  $T_i$  of fixed size  $s_i \times s_i$  where  $r$  is the number of conjugacy classes of  $G$  and the  $s_i$  are the ranks of the group ring matrices of the primitive idempotents. Thus the group ring  $\mathbb{C}G$  is isomorphic to matrices of the type  $\text{diag}(T_1, T_2, \dots, T_r)$ . A faithful representation of the group itself may be given by taking images of the group elements. Other representations of  $G$  may be obtained using the blocks in the images of the of the group elements.

See Sections 4 and 5 below for applications and examples; these show how interesting and useful representations of the groups, and group rings, may be derived by the method.

The finite abelian group ring is a special case but is dealt with independently in Section 5 as more direct information and direct calculations may be made. The diagonalising matrix is obtained directly from Fourier type matrices, the diagonal entries are obtained from the entries of the first row of the group ring matrix and the character table may be read off from the diagonalising matrix.

More precisely, for a given finite abelian group  $H$  it is shown explicitly that exists a matrix  $P$  such that  $P^{-1}BP$  is diagonal where  $B$  is any group ring matrix of  $H$ . The matrix  $P$  is independent of the entries of the particular group ring matrix  $B$  and the diagonal entries are given precisely in terms of the entries of the first row of  $B$ . The matrix  $P$  may be chosen so that  $PP^* = nI_n$  and when the rows of  $P$  are labelled appropriately to the structure of the group as a product of cyclic groups, then the rows of  $P$  give the characters and character table of  $H$ .

Many results for circulant  $n \times n$  matrices (= group ring matrices over the cyclic group  $C_n$ ) hold not just over  $\mathbb{C}$  but over any field  $F$  which contains a primitive  $n^{\text{th}}$  root of unity. Similarly some results here hold over fields other than  $\mathbb{C}$  but this aspect is not dealt with here.

The idea of using group ring matrices of complete orthogonal sets of idempotents originated in [5] where these are used in the study and construction of types of multidimensional paraunitary matrices.

## 2 Group ring matrices

Certain classes of matrices are determined by their first row or column. A particular type of such matrices are those corresponding to group rings. It is shown in [4] that the group ring  $RG$  where  $|G| = n$  may be embedded in the ring of  $n \times n$  matrices over  $R$  in a precise manner.

Let  $\{g_1, g_2, \dots, g_n\}$  be a fixed listing of the elements of  $G$ . Consider the following matrix:

$$\begin{pmatrix} g_1^{-1}g_1 & g_1^{-1}g_2 & g_1^{-1}g_3 & \cdots & g_1^{-1}g_n \\ g_2^{-1}g_1 & g_2^{-1}g_2 & g_2^{-1}g_3 & \cdots & g_2^{-1}g_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_n^{-1}g_1 & g_n^{-1}g_2 & g_n^{-1}g_3 & \cdots & g_n^{-1}g_n \end{pmatrix}$$

Call this the *matrix of  $G$*  (relative to this listing) and denote it by  $M(G)$ .

### 2.1 $RG$ -matrix

Given a listing of the elements of  $G$ , form the matrix  $M(G)$  of  $G$  relative to this listing. An  $RG$ -matrix over a ring  $R$  is a matrix obtained by substituting elements of  $R$  for the elements of  $G$  in  $M(G)$ . If

$w \in RG$  and  $w = \sum_{i=1}^n \alpha_i g_i$  then  $\sigma(w)$  is the  $n \times n$   $RG$ -matrix obtained by substituting each  $\alpha_i$  for  $g_i$  in the group matrix.

$$\text{Precisely } \sigma(w) = \begin{pmatrix} \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \alpha_{g_1^{-1}g_3} & \cdots & \alpha_{g_1^{-1}g_n} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \alpha_{g_2^{-1}g_3} & \cdots & \alpha_{g_2^{-1}g_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{g_n^{-1}g_1} & \alpha_{g_n^{-1}g_2} & \alpha_{g_n^{-1}g_3} & \cdots & \alpha_{g_n^{-1}g_n} \end{pmatrix}$$

It is shown in [4] that  $w \mapsto \sigma(w)$  gives an isomorphism of the group ring  $RG$  into the ring of  $n \times n$  matrices over  $R$ .

Given the entries of the first row of an  $RG$ -matrix the entries of the other rows are determined from the matrix  $M(G)$  of  $G$ .

An  $RG$ -matrix is a matrix corresponding to a group ring element in the isomorphism from the

group ring into the ring of  $R_{n \times n}$  matrices. The isomorphism depends on the listing of the elements of  $G$ . For example if  $G$  is cyclic, an  $RG$ -matrix is a circulant matrix relevant to the natural listing  $G = \{1, g, g^2, \dots, g^{n-1}\}$  where  $G$  is generated by  $g$ . An  $RG$ -matrix when  $G$  is dihedral is one of the form  $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$  where  $A$  is circulant and  $B$  is reverse circulant but also one of the form  $\begin{pmatrix} A & B \\ B^T & A^T \end{pmatrix}$  where both  $A, B$  are circulant, in a different listing of  $G$ . Other examples are given within [4].

In general given a group ring element  $w$ , and a fixed listing of the elements of the group, the corresponding capital letter  $W$  is often used to denote the image of  $w$ ,  $\sigma(w)$ , in the ring of  $RG$ -matrices.

### Listing

Changing the listing of the elements of the group gives an equivalent  $RG$ -matrix and one is obtained from the other by a sequence of processes consisting of interchanging two rows and then interchanging the corresponding two columns.

## 3 Block diagonal

Matrices, when diagonalisable, may be simultaneously diagonalised if and only if they commute. However a set of matrices may be simultaneously block diagonalisable in the sense that there exist a matrix  $U$  such that  $U^{-1}AU$  has the form  $\text{diag}(T_1, T_2, \dots, T_r)$ , where each  $T_i$  is of fixed  $r_i \times r_i$  size, for every matrix  $A$  in the set – and for  $U$  independent of  $A$ . This is the case for group ring matrices.

Idempotents will naturally play an important part. (See [5] where these are used for paraunitary matrices.) Say  $e$  is an idempotent in a ring  $R$  if  $e^2 = e$  and say  $\{e, f\}$  are orthogonal if  $ef = 0 = fe$ . Say  $\{e_1, e_2, \dots, e_k\}$  is a complete orthogonal set of idempotents in a ring  $R$  if  $e_i^2 = e_i, e_i e_j = 0$  for  $i \neq j$  and  $e_1 + e_2 + \dots + e_k = 1$  where 1 is the identity of  $R$ . Now  $\text{tr}A$  denotes the trace of a matrix  $A$ .

**Proposition 3.1** *Suppose  $\{e_1, e_2, \dots, e_k\}$  is a complete orthogonal set of idempotents. Consider  $w = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_k e_k$  with  $\alpha_i \in F$ , a field. Then  $w$  is invertible if and only if each  $\alpha_i \neq 0$  and in this case  $w^{-1} = \alpha_1^{-1} e_1 + \alpha_2^{-1} e_2 + \dots + \alpha_k^{-1} e_k$ .*

**Proof:** Suppose each  $\alpha_i \neq 0$ . Then  $w * (\alpha_1^{-1} e_1 + \alpha_2^{-1} e_2 + \dots + \alpha_k^{-1} e_k) = e_1^2 + e_2^2 + \dots + e_k^2 = e_1 + e_2 + \dots + e_k = 1$ .

Suppose  $w$  is invertible and that some  $\alpha_i = 0$ . Then  $w e_i = 0$  and so  $w$  is a (non-zero) zero-divisor and is not invertible.  $\square$

**Lemma 3.1** *Let  $\{E_1, E_2, \dots, E_s\}$  be a set of orthogonal idempotent matrices. Then  $\text{rank}(E_1 + E_2 + \dots + E_s) = \text{tr}(E_1 + E_2 + \dots + E_s) = \text{tr}E_1 + \text{tr}E_2 + \dots + \text{tr}E_s = \text{rank}E_1 + \text{rank}E_2 + \dots + \text{rank}E_s$ .*

**Proof:** It is known that  $\text{rank}A = \text{tr}A$  for an idempotent matrix, see for example [1], and so  $\text{rank}E_i = \text{tr}E_i$  for each  $i$ . If  $\{E, F, G\}$  is a set of orthogonal idempotent matrices so is  $\{E + F, G\}$ . From this it follows that  $\text{rank}(E_1 + E_2 + \dots + E_s) = \text{tr}(E_1 + E_2 + \dots + E_s) = \text{tr}E_1 + \text{tr}E_2 + \dots + \text{tr}E_s = \text{rank}E_1 + \text{rank}E_2 + \dots + \text{rank}E_s$ .  $\square$

**Corollary 3.1**  $\text{rank}(E_{i_1} + E_{i_2} + \dots + E_{i_k}) = \text{rank}E_{i_1} + \text{rank}E_{i_2} + \dots + \text{rank}E_{i_k}$  for  $i_j \in \{1, 2, \dots, s\}$ ,  $i_j \neq i_l$  for  $j \neq l$ .

Let  $A = a_1 E_1 + a_2 E_2 + \dots + a_k E_k$  for a complete set of idempotent orthogonal matrices  $E_i$ . Then  $A$  is invertible if and only if each  $a_i \neq 0$  and in this case  $A^{-1} = a_1^{-1} E_1 + a_2^{-1} E_2 + \dots + a_k^{-1} E_k$ . This is a special case of the following.

**Proposition 3.2** *Suppose  $\{E_1, E_2, \dots, E_k\}$  is a complete symmetric orthogonal set of idempotents in  $F_{n \times n}$ . Let  $Q = a_1 E_1 + a_2 E_2 + \dots + a_k E_k$ . Then the determinant of  $Q$  is  $|Q| = a_1^{\text{rank}E_1} a_2^{\text{rank}E_2} \dots a_k^{\text{rank}E_k}$ .*

Let  $RG$  be the group ring of a finite group  $G$  over the ring  $R$ . Let  $\{e_1, e_2, \dots, e_k\}$  be a complete orthogonal set of idempotents in  $RG$  and  $\{E_1, E_2, \dots, E_k\}$  the corresponding  $RG$ -matrices (relevant to

some listing of the elements of  $G$ ). Such a set of idempotents is known to exist when  $R = \mathbb{C}$ , the complex numbers, and also over other fields, see for example [3] or [7]. We will confine ourselves here to  $\mathbb{C}$  but many of the results hold over these other fields. The idempotent elements from the group ring satisfy  $e^* = e$  and so the idempotent matrices are symmetric,  $E^* = E$ , and satisfy  $E^2 = EE^* = E$ .

We now specialise the  $E_i$  to be  $n \times n$  matrices corresponding to the group ring idempotents  $e_i$ , that is  $\sigma e_i = E_i$ . Define the rank of  $e_i$  to be that of  $E_i$ .

Consider now the group ring  $FG$  where  $F = \mathbb{C}$  the complex numbers and  $G$  is a finite group. As already mentioned,  $FG$  contains a complete orthogonal set of idempotents  $\{e_1, e_2, \dots, e_k\}$  which may be taken to be primitive, [7].

**Theorem 3.1** *Let  $A$  be a  $FG$ -matrix with  $F = \mathbb{C}$ . Then there exists a non-singular matrix  $P$  independent of  $A$  such that  $P^{-1}AP = T$  where  $T$  is a block diagonal matrix with blocks of size  $r_i \times r_i$  for  $i = 1, 2, \dots, k$  and  $r_i$  are the ranks of the  $e_i$ .*

**Proof:** Let  $\{e_1, e_2, \dots, e_k\}$  be the orthogonal idempotents and  $S = \{E_1, E_2, \dots, E_k\}$  the group ring matrices corresponding to these, that is,  $\sigma(e_i) = E_i$  in the embedding of the group ring into the  $CG$ -matrices. Any column of  $E_i$  is orthogonal to any column of  $E_j$  for  $i \neq j$  as  $E_i E_j^* = 0$ . Now let  $\text{rank } E_i = r_i$ . Then  $\sum_{i=1}^k r_i = n$ . Let  $S_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,r_i}\}$  be a basis for the column space of  $E_i$  consisting of a subset of the columns of  $E_i$ ; do this for each  $i$ . Then each element of  $S_i$  is orthogonal to each element of  $S_j$  for  $i \neq j$ . Since  $\sum_{i=1}^k r_i = n$  it follows that  $S = \{S_1, S_2, \dots, S_k\}$  is a basis for  $F^n$ .

Let  $V_{i,j}$  denote the  $FG$ -matrix determined by the column vector  $v_{i,j}$ , let  $S_i(G)$  denote the set of  $FG$ -matrices obtained by substituting  $V_{i,j}$  for  $v_{i,j}$  in  $S_i$  and let  $S(G)$  denote the set of  $FG$ -matrices obtained by substituting  $S_i(G)$  for  $S_i$  in  $S$ .

As  $S$  is a basis for  $F^n$  the first column of  $AE_i$  is a linear combination of elements from  $S$ . The first column of  $AE_i$  determines  $AE_i$ , as  $AE_i$  is an  $FG$ -matrix, and hence  $AE_i$  is a linear combination of elements of  $S(G)$ . By multiplying  $AE_i$  through on the right by  $E_i$ , and orthogonality, it follows that  $AE_i$  is a linear combination of  $S_i(G) = \{V_{i,1}, V_{i,2}, \dots, V_{i,r_i}\}$ . Now each  $V_{i,j}$  consists of columns which are a permutation of the columns of  $E_i$ . Also  $E_i$  contains the columns  $S_i$ . Thus equating  $AE_i$  to the linear combination of  $S_i(G)$  implies that each  $AV_{i,j}$  is a linear combination of  $S_i$ .

Let  $P$  then be the matrix with columns consisting of the first columns  $S_i$  for  $i = 1, 2, \dots, k$ . Then  $AP = PA$  where  $T$  is a matrix of blocks of size  $r_i \times r_i$  arranged diagonally for  $i = 1, 2, \dots, k$ . Since  $P$  is invertible it follows that  $P^{-1}AP = T$ . □

The proof is constructive in the sense that the matrix  $P$  is constructed from the complete orthogonal set of idempotents. Method:

1. Find complete orthogonal set of idempotents  $\{e_1, e_2, \dots, e_k\}$  for  $FG$ .
2. Construct the corresponding  $FG$ -matrices  $\{E_1, E_2, \dots, E_k\}$ .
3. Find a basis  $S_i$  for the column space of  $E_i$  for  $1 \leq i \leq k$ .
4. Let  $P$  be the matrix made up of columns of the union of the  $S_i$ .
5. Then  $P^{-1}AP$  is a block diagonal matrix consisting of blocks of size  $r_i \times r_i$  where  $r_i$  is the rank of  $E_i$ .

However this algorithm requires being able to construct a complete orthogonal set of idempotents. If the matrix  $P$  could be obtained directly then indeed this would be a way for the construction of the idempotents.

**Corollary 3.2** *The group ring  $FG$  is isomorphic to a subring of such block diagonal matrices. The isomorphism is given by  $w \mapsto \sigma(w) = W \mapsto P^{-1}WP$ .*

The isomorphism includes an isomorphic embedding of the group  $G$  itself into the set of such block diagonal matrices. Other linear representations of  $G$  may be obtained by using the block images of the group elements.

**Theorem 3.2** Suppose  $A$  is an  $FG$ -matrix where  $F = \mathbb{C}$ . Then there exists a unitary matrix  $P$  such that  $P^T A P = T$  where  $T$  is a block diagonal matrix with blocks of size  $r_i \times r_i$  for  $i = 1, 2, \dots, k$  along the diagonal.

**Proof:** The diagonalising matrix in the proof of Theorem 3.1 may be made unitary by constructing an orthonormal basis for space generated by  $\{V_{i,1}, V_{i,2}, \dots, V_{i,r_i}\}$  for each  $i = 1, 2, \dots, k$ . Let  $S_i = \{W_{i,1}, W_{i,2}, \dots, W_{i,r_i}\}$  be an orthonormal basis for the space spanned by  $\{V_{i,1}, V_{i,2}, \dots, V_{i,r_i}\}$ . Then  $\hat{S} = \{S_1, S_2, \dots, S_k\}$  is an orthonormal basis for  $F^n$ . Set  $P$  to be the matrix with elements of  $\hat{S}$  as columns. Then  $P$  is unitary and  $P^T A P = T$  as required.  $\square$

The group ring is isomorphic to the ring of  $RG$ -matrices, [4], and the ring of  $RG$  matrices is isomorphic to the ring of such block diagonal matrices under the mapping  $w \mapsto \sigma(w) = W \mapsto P^{-1} W P$  for this fixed  $P$ .

## 4 Cases, applications

See for example [3, 6, 7] for information on representation theory including characters and character tables. See for example [4] for information on group ring matrices and in particular on the method for obtaining the corresponding group ring matrix from a group ring element.

### 4.1 Abelian

When  $G = C_n$ , the cyclic group of order  $n$ , the matrix  $P$  of Theorem 3.1 is the Fourier matrix and  $T$  is a diagonal matrix. The case when  $G$  is any abelian group is dealt with fully in section 5.

### 4.2 Dihedral

The dihedral group  $D_{2n}$  is generated by elements  $a$  and  $b$  with presentation:

$$\langle a, b \mid a^n = 1, b^2 = 1, bab = a^{-1} \rangle$$

It has order  $2n$ , and a natural listing of the elements is  $\{1, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b\}$ .

As every element in  $D_{2n}$  is conjugate to its inverse, the complex characters of  $D_{2n}$  are real. The characters  $D_{2n}$  are contained in an extension of  $\mathbb{Q}$  of degree  $\phi(n)/2$  and this is  $\mathbb{Q}$  only for  $2n \leq 6$ . Here  $\phi$  is the Euler phi function. Let  $S_n$  denote the symmetric group of order  $n$ . The characters of  $S_n$  are rational.

#### 4.2.1 $S_3 = D_6$

Consider  $D_6$ . Note that  $D_6 = S_3$ . The conjugacy classes are  $\{1\}, \{a, a^2\}, \{b, ab, ab^2\}$ . The central (primitive, symmetric) idempotents are  $e_0 = 1/6(1 + a + a^2 + b + ba + ba^2)$ ,  $e_1 = 1/6(1 + a + a^2 - b - ba - ba^2)$ ,  $e_3 = 1/3(2 - a - a^2)$ .

This gives the corresponding group ring matrices:

$$E_0 = \frac{1}{6} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, E_1 = \frac{1}{6} \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \end{pmatrix}, E_2 = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}.$$

Now  $E_0, E_1$  have rank 1 and  $E_2$  has rank 4 from general theory.

Thus we need a set consisting of one column from each of  $E_0, E_1$  and 4 linearly independent columns from  $E_2$  to form a set of 6 linearly independent vectors. It is easy to see that

$$v_1 = (1, 1, 1, 1, 1, 1)^T, v_2 = (1, 1, 1, -1, -1, -1)^T, v_3 = (2, -1, -1, 0, 0, 0)^T, \\ v_4 = (-1, 2, -1, 0, 0, 0)^T, v_5 = (0, 0, 0, 2, -1, -1)^T, v_6 = (0, 0, 0, -1, 2, -1)^T$$

is such a set.

Now let  $P = (v_1, v_2, v_3, v_4, v_5, v_6)$ . Then for any  $\mathbb{C}D_6$  matrix  $A$ ,  $P^{-1} A P = \text{diag}(a, b, D)$  where  $D$  is a  $4 \times 4$  matrix.

As noted, the group ring is isomorphic to the ring of  $RG$ -matrices, and the ring of  $RG$ -matrices is isomorphic to the ring of such block diagonal matrices under the mapping  $A \mapsto P^{-1}AP$  for this fixed  $P$  (Theorem 3.1).

Now consider the image of  $D_6$  itself under this isomorphism. The matrix  $A$  of  $a \in S_3 = D_6$  in this isomorphism is mapped to  $P^{-1}AP$ . Here  $A = \left( \begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)$

$$\text{and } P^{-1}AP = \left( \begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array} \right).$$

(In some cases it is easier to work out  $AP$  and then solve for  $D$  in  $PD$  where  $D$  is of the correct block diagonal type.)

$$\text{Similarly the image of } b \text{ is obtained; } B \text{ is the } RG\text{-matrix of } b \text{ and } P^{-1}BP = \left( \begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right).$$

Representations of  $S_3 = D_6$  may be obtained using the block matrices of the images of the group elements. For example  $a \mapsto \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, b \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$  gives a representation of  $D_6 = S_3$ .

It may be shown directly from the structure of  $P$  and of  $A$  corresponding to a group element  $a$  that the  $4 \times 4$  part in  $P^{-1}AP$  has the form  $T = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$  or of the form  $S = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$  where  $X, Y$  are  $2 \times 2$  blocks.

Say a matrix is in  $\mathbb{T}$  if it has the form  $T$  and say a matrix is in  $\mathbb{S}$  if it has the form  $S$ . Interestingly then generally  $TS \in \mathbb{S}, ST \in \mathbb{S}, T_1T_2 \in \mathbb{T}, S_1S_2 \in \mathbb{T}$ , for  $S, S_1, S_2 \in \mathbb{S}, T, T_1, T_2 \in \mathbb{T}$ .

#### 4.2.2 Unitary required?

Now  $P$  may be made orthogonal by finding an orthogonal basis for the 4 linearly independent columns of  $E_2$  and then dividing each of the resulting set of 6 vectors by their lengths.

An orthogonal basis for the columns of  $E_2$  is

$$\{(2, -1, -1, 0, 0, 0)^T, (0, 1, -1, 0, 0, 0)^T, (0, 0, 0, 2, -1, -1)^T, (0, 0, 0, 0, 1, -1)^T\}.$$

Construct an orthonormal basis:

$$v_1 = \sqrt{\frac{1}{6}}(1, 1, 1, 1, 1, 1)^T, v_2 = \sqrt{\frac{1}{6}}(1, 1, 1, -1, -1, -1)^T, v_3 = \sqrt{\frac{1}{6}}(2, -1, -1, 0, 0, 0)^T, \\ v_4 = \sqrt{\frac{1}{2}}(0, 1, -1, 0, 0, 0)^T, v_5 = \sqrt{\frac{1}{6}}(0, 0, 0, 2, -1, -1)^T, v_6 = \sqrt{\frac{1}{2}}(0, 0, 0, 0, 1, -1)^T.$$

Now construct the unitary (orthogonal in this case) matrix  $P = (v_1, v_2, v_3, v_4, v_5, v_6)$ . Then for any  $\mathbb{C}D_6$  matrix  $A$ ,  $P^*AP = \text{diag}(a, b, D)$  where  $D$  is a  $4 \times 4$  matrix.

When  $P$  is unitary, = orthogonal in this case, then  $P^TAP$  and  $P^TBP$  are unitary as  $A, B$  are orthogonal. The diagonal  $4 \times 4$  matrix must then be orthogonal. For example:

$$P^T AP = P^* AP = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & \sqrt{3}/2 & 0 & 0 \\ 0 & 0 & -\sqrt{3}/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & -\sqrt{3}/2 \\ 0 & 0 & 0 & 0 & \sqrt{3}/2 & -1/2 \end{pmatrix}$$

The  $4 \times 4$  block matrix is easily checked to be unitary/orthogonal as expected from theory.

### 4.3 Other dihedral

The character tables for  $D_{2n}$  may be derived from [3, 6] and are also available at various on-line resources such as that of Jim Belk. We outline how the results may be applied in the case of  $D_{10}$ .

The character table of  $D_{10}$  is the following:

$$\begin{pmatrix} 1 & b & a & a^2 \\ 1 & 5 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 2 & 0 & 2 \cos(2\pi/5) & 2 \cos(4\pi/5) \\ 2 & 0 & 2 \cos(4\pi/5) & 2 \cos(8\pi/5) \end{pmatrix}.$$

This gives the following complete (symmetric) orthogonal set of idempotents in the group ring:  $e_0 = \frac{1}{10}(1+a+a^2+a^3+a^4+b+ba+ba^2+ba^3+ba^4)$ ,  $e_1 = \frac{1}{10}(1+a+a^2+a^3+a^4-b-ba-ba^2-ba^3-ba^4)$ ,  $e_2 = \frac{4}{10}(1+\cos(2\pi/5)a+\cos(4\pi/5)a^2+\cos(4\pi/5)a^3+\cos(2\pi/5)a^4)$ ,  $e_3 = \frac{4}{10}(1+\cos(4\pi/5)a+\cos(8\pi/5)a^2+\cos(8\pi/5)a^3+\cos(4\pi/5)a^4)$ .

Let  $\sigma(e_i) = E_i$  – this is the image of the group ring element  $e_i$  in the group ring matrix. Each of  $E_0, E_1$  has rank 1 and each of  $E_2, E_3$  has rank 4. Four linearly independent columns in each of  $E_2, E_3$  are easy to obtain and indeed four orthogonal such may be derived if required. The matrix  $P$  is formed using the first columns of  $E_1, E_2$  and 4 linearly independent columns of each of  $E_3$  and  $E_4$ . Then  $P^{-1}AP = \text{diag}(\alpha_1, \alpha_2, T_1, T_2)$  for any group ring matrix  $A$  of  $D_{10}$  where  $T_1, T_2$  are  $4 \times 4$  block matrices. Then the composition of mappings  $w \mapsto \sigma(w) = W \mapsto P^{-1}WP$  is an isomorphism. Representations of the group may be obtained by specialising to blocks of the images of the group elements.

The form of  $P$  is  $\begin{pmatrix} A & C & 0 & D & 0 \\ B & 0 & C_1 & 0 & D_1 \end{pmatrix}$  for suitable  $5 \times 2$  blocks  $A, 0, C, D, C_1, D_1$ . Then it may be shown that in  $P^{-1}AP$  the two  $4 \times 4$  blocks have the form  $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$  or else the form  $\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$  for  $2 \times 2$  blocks  $X, Y$  when  $A$  corresponds to a group element  $a$ .

### 4.4 Quaternion group of order 8

The five primitive central idempotents  $\{e_1, e_2, e_3, e_4, e_5\}$  of  $\mathbb{C}K_8$  where  $K_8$  is the quaternion group of order 8 is given in [7] page 186.  $K_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \rangle$  and is listed as  $\{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$ .

$$\begin{aligned} e_1 &= 1/8(1+a+a^2+a^3+b+ab+a^2b+a^3b) \\ e_2 &= 1/8(1+a+a^2+a^3-b-ab-a^2b-a^3b) \\ e_3 &= 1/8(1-a+a^2-a^3+b-ab+a^2b-a^3b) \\ e_4 &= 1/8(1-a+a^2-a^3-b+ab-a^2b+a^3b) \\ e_5 &= 1/2(1-a^2) \end{aligned}$$

([7] has  $-ab$  in  $e_4$  which should be  $+ab$  as above.)

The group ring matrices  $\{E_1, E_2, E_3, E_4\}$  corresponding to  $\{e_1, e_2, e_3, e_4\}$  respectively have rank 1 and the group ring matrix  $E_5$  corresponding to  $e_5$  has rank 4, which can be seen from theory. Thus take the first columns of  $E_1, E_2, E_3, E_4$  and 4 linearly independent columns of  $E_5$  to form a matrix  $P$ . Then  $P^{-1}AP = \text{diag}(T_1, T_2, T_3, T_4, T_5)$  where  $T_1, T_2, T_3, T_4$  are scalars and  $T_5$  is a  $4 \times 4$  matrix, for any group ring matrix  $A$  of  $K_8$ .

Precisely we may take:

$$P = \left( \begin{array}{c|c|c|c|c|c|c|c} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 & 0 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & -1 \end{array} \right)$$

and then  $P^{-1}AP = \text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, T)$  for any group ring matrix  $A$  of  $K_8$  where  $T$  is a  $4 \times 4$  matrix.

This gives an isomorphism from the group ring of  $K_8$  to these block matrices given by  $w \mapsto \sigma(w) = W \mapsto P^{-1}WP$ . Representations of  $K_8$  may be obtained by specialising to the group elements.

The following then gives an embedding of  $K_8$ :

$$a \mapsto \left( \begin{array}{c|c} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \right), b \mapsto \left( \begin{array}{c|c} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \end{array} \right)$$

Using the blocks gives other representations. For example

$$a \mapsto \left( \begin{array}{c|c} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{array} \right), b \mapsto \left( \begin{array}{c|c} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \right)$$

gives a representation of  $K_8$ .

It may be shown directly from the block form of  $P$  that the image of a group element has the  $4 \times 4$  block of the form  $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$  or else the form  $\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$  for  $2 \times 2$  blocks  $X, Y$ .

## 5 Abelian groups

The abelian group case follows from the general case, Section 3, but may be tackled directly and more illuminatingly as follows.

Let  $\{A_1, A_2, \dots, A_k\}$  be an ordered set of matrices of the same size. Then the *block circulant matrix* formed from the set is  $A = \text{circ}(A_1, A_2, \dots, A_k) = \begin{pmatrix} A_1 & A_2 & \dots & A_k \\ A_k & A_1 & \dots & A_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \dots & A_1 \end{pmatrix}$

If the  $A_i$  have size  $m \times t$  then  $A$  has size  $km \times kt$ . The block circulant formed depends on the order of the elements in  $\{A_1, A_2, \dots, A_k\}$ .

Let  $P$  be an  $n \times n$  matrix. Then the *block Fourier matrix*  $P_f$  corresponding to  $P$  is  $P \otimes F$ , the tensor product of  $P$  and  $F$  where  $F$  is the Fourier  $n \times n$  matrix.

$$\text{Thus } P_f = P \otimes F = \begin{pmatrix} P & P & P & \dots & P \\ P & \omega P & \omega^2 P & \dots & \omega^{n-1} P \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P & \omega^{n-1} P & \omega^{2(n-1)} & \dots & \omega^{(n-1)(n-1)} P \end{pmatrix}$$

It is clear then that:

**Proposition 5.1**  $P_f$  is invertible if and only if  $P$  is invertible and the inverse when it exists is  $P^{-1} \otimes F^*$ .

Here  $F^*$  denotes the inverse of the Fourier matrix. If the Fourier matrix is normalised in  $\mathbb{C}$ , then  $F^*$



is the complex conjugate transposed of  $F$ .

The following theorem may be proved in a manner similar to the proof that the Fourier matrix diagonalises a circulant matrix.

**Theorem 5.1** Suppose  $\{A_1, A_2, \dots, A_k\}$  are matrices of the same size and can be simultaneously diagonalised by  $P$  with  $P^{-1}A_iP = D_i$  where each  $D_i$  is diagonal. Then the block circulant matrix  $A$  formed

$$from these matrices can be diagonalised by  $P_f = P \otimes F = \begin{pmatrix} P & P & P & \dots & P \\ P & \omega P & \omega^2 P & \dots & \omega^{k-1} P \\ P & \omega^2 P & \omega^4 P & \dots & \omega^{2(k-1)} P \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P & \omega^{k-1} P & \omega^{2(k-1)} P & \dots & \omega^{(k-1)(k-1)} P \end{pmatrix}$$$

where  $\omega$  is a primitive  $k^{th}$  root of unity.

Moreover  $P_f^{-1}AP_f = D$  where  $D$  is diagonal and

$$D = \text{diag}(D_1 + D_2 + \dots + D_k, D_1 + \omega D_2 + \dots + \omega^{k-1} D_k, D_1 + \omega^2 D_2 + \omega^4 D_3 + \dots + \omega^{2(k-1)} D_k, \dots, D_1 + \omega^{k-1} D_2 + \omega^{2(k-1)} D_3 + \dots + \omega^{(k-1)(k-1)} D_k)$$

**Proof:**

The proof of this is direct, involving working out  $AP_f$  and showing it is  $P_f D$ , with  $D$  as given. Since  $P_f$  is invertible by Proposition 5.1 the result will follow. This is similar to a proof that the Fourier matrix diagonalises a circulant matrix. □

The simultaneous diagonalisation process of the Theorem may then be repeated.

Suppose now  $G = K \times H$ , the direct product of  $K, H$ , and  $H$  is cyclic. Then a group ring matrix of  $G$  is of the form  $M = \text{circ}(K_1, K_2, \dots, K_h)$  where  $K_i$  are group ring matrices of  $K$  and  $|H| = h$ . If the  $K_i$  can be diagonalised by  $P$  then  $M$  can be diagonalised by the Fourier block matrix formed from  $P$  by Theorem 5.1. A finite abelian group is the direct product of cyclic groups and thus repeating the process enables the simultaneous diagonalisation of the group ring matrices of a finite abelian group and it gives an explicit diagonalising matrix. The characters and character table of the finite abelian group may be read off from the diagonalising matrix.

Since the Fourier  $n \times n$  matrix diagonalises a circulant  $n \times n$  matrix, and the Fourier matrix is a Hadamard complex matrix, the diagonalising matrix  $P$  of size  $q \times q$ , constructed by iteration of Theorem 5.1, of a group ring matrix of a finite abelian group is then seen to satisfy  $PP^* = qI$  and to have roots of unity as entries. It is thus a special type of Hadamard complex matrix.

The examples below illustrate the method.

## 5.1 Examples

- Consider  $G = C_3 \times C_3$ . Now  $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$  where  $\omega$  is a primitive  $3^{rd}$  root of unity

diagonalises any circulant  $3 \times 3$  matrix which is the group ring matrix of  $C_3$ . Then  $P_f =$

$$\begin{pmatrix} P & P & P \\ P & \omega P & \omega^2 P \\ P & \omega^2 P & \omega P \end{pmatrix} \text{ diagonalises any group ring matrix of } C_3 \times C_3.$$

Written out in full:  $P_f = \left( \begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 & \omega^2 & \omega \\ \hline 1 & 1 & 1 & \omega & \omega & \omega & \omega^2 & \omega^2 & \omega^2 \\ 1 & \omega & \omega^2 & \omega & \omega^2 & 1 & \omega^2 & 1 & \omega \\ 1 & \omega^2 & \omega & \omega & 1 & \omega^2 & \omega^2 & \omega & 1 \\ \hline 1 & 1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega & \omega & \omega \\ 1 & \omega & \omega^2 & \omega^2 & 1 & \omega & \omega & \omega^2 & 1 \\ 1 & \omega^2 & \omega & \omega^2 & \omega & 1 & \omega & 1 & \omega^2 \end{array} \right)$

The characters and character table of  $C_3 \times C_3$  may be read off from the rows of  $P_f$  by labelling the rows of  $P_f$  appropriate to the listing of the elements of  $C_3 \times C_3$  when forming the group ring matrices. The listing here is  $\{1, g, g^2, h, hg, hg^2, h^2, h^2g, h^2g^2\}$  where the  $C_3$  are generated by  $\{g, h\}$  respectively. Thus the character table of  $C_3 \times C_3$  is

$$\begin{pmatrix} 1 & g & g^2 & h & hg & hg^2 & h^2 & h^2g & h^2g^2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 & \omega^2 & \omega \\ 1 & 1 & 1 & \omega & \omega & \omega & \omega^2 & \omega^2 & \omega^2 \\ 1 & \omega & \omega^2 & \omega & \omega^2 & 1 & \omega^2 & 1 & \omega \\ 1 & \omega^2 & \omega & \omega & 1 & \omega^2 & \omega^2 & \omega & 1 \\ 1 & 1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega & \omega & \omega \\ 1 & \omega & \omega^2 & \omega^2 & 1 & \omega & \omega & \omega^2 & 1 \\ 1 & \omega^2 & \omega & \omega^2 & \omega & 1 & \omega & 1 & \omega^2 \end{pmatrix}$$

Note that  $\frac{1}{\sqrt{9}}P_f$  is unitary and that  $P_f$  is a Hadamard complex matrix.

- For  $C_2 \times C_4$  consider  $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$  and note that a primitive  $4^{th}$  root of 1 is  $i = \sqrt{-1}$ . Then

$$i^2 = -1, i^3 = -i, i^4 = 1. \text{ Now form } Q = \begin{pmatrix} P & P & P & P \\ P & iP & -P & -iP \\ P & -P & P & -P \\ P & -iP & -P & iP \end{pmatrix}. \text{ The characters of } C_2 \times C_4 \text{ can}$$

be read off from  $Q$ ,  $Q$  is a Hadamard complex matrix and  $\frac{1}{\sqrt{8}}Q$  is unitary.

- For  $C_3 \times C_4$  consider that  $C_3 \times C_4 \cong C_{12}$ . Then the diagonalising matrix obtained using the natural ordering in  $C_3 \times C_4$  is equivalent to the diagonalising matrix using the natural ordering in  $C_{12}$ .
- Consider  $C_2^n$ . Let  $P_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$  and inductively define for  $n \geq 2$ ,  $P_n = \begin{pmatrix} P_{n-1} & P_{n-1} \\ P_{n-1} & -P_{n-1} \end{pmatrix}$ .

Then  $P_n$  diagonalises any  $\mathbb{C}C_2^n$ -matrix and the characters of  $C_2^n$  may be read off from the rows of  $P_n$ . Note that  $P_n$  is a Hadamard (real) matrix.

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